

# Thermodynamics on the apparent horizon in generalized gravity theories

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We present a general procedure to construct the first law of thermodynamics on the apparent horizon and illustrate its validity by examining it in some extended gravity theories. Applying this procedure, we can describe the thermodynamics on the apparent horizon in Randall-Sundrum braneworld imbedded in a nontrivial bulk. We discuss the mass-like function which was used to link Friedmann equation to the first law of thermodynamics and obtain its special case which gives the generalized Misner-Sharp mass in Lovelock gravity.

PACS numbers: 04.70.Dy, 04.50.-h, 98.80.-k

## I. INTRODUCTION

Inspired by black hole thermodynamics, it was realized that there is a profound connection between gravity and thermodynamics. Jacobson first showed that the Einstein gravity can be derived from the first law of thermodynamics in the Rindler spacetime. For a general static spherically symmetric spacetime, Padmanabhan pointed out that Einstein equations at the horizon give rise to the first law of thermodynamics [2]. Recently the study on the connection between gravity and thermodynamics has been extended to cosmological context. Frolov and Kofman [3] employed the approach proposed by Jacobson [1] to a quasi-de Sitter geometry of inflationary universe, and calculated the energy flux of a background slow-roll scalar through the quasi-de Sitter apparent horizon. By applying the first law of thermodynamics to a cosmological horizon, Danielsson obtained Friedmann equation in the expanding universe [4]. In the quintessence dominated accelerating universe, Bousso [5] showed that the first law of thermodynamics holds at the apparent horizon. Cai and Kim [6] generalized the derivation of the Friedmann equations from the first law of thermodynamics to the spacetime with any spatial curvature. This study has also been generalized to the  $f(R)$  gravity [7, 8] and scalar-tensor gravity theory [9].

Besides gravity theories in four dimensions, the study on the connection between gravity and thermodynamics has also been extended to the braneworld cosmology [10, 11, 12, 13]. It has been suggested that the first law of thermodynamics on the apparent horizon can be derived from the Friedmann equations in the Randall-Sundrum braneworld [10, 11], also in braneworld with curvature corrections including 4D scalar curvature from induced gravity on the brane and the 5D Gauss-Bonnet curvature correction [12], and the braneworld with finite brane thickness [13]. In the braneworld the exact black hole solution has not been discovered until now. It was pointed out that the connection between gravity and thermodynamics can shed lights on the entropy of the braneworld [11, 12].

It is still unclear whether the connection between gravity and thermodynamics also holds in more general gravity theories. It was argued that if one does not know the bulk geometry, it is not easy to build up thermodynamics in such spacetime [10]. However, if one looks at  $f(R)$  gravity, although the concrete form in this nonlinear gravity theory is not available, the corresponding thermodynamics can still be obtained, with the price of introducing the entropy production. In this work we are going to investigate this problem. We will present a general procedure to construct thermodynamics on the apparent horizon and show that this procedure correctly works, examined in some extended gravity theories. We will try to explain the reason that the additional entropy production terms appear in the  $f(R)$  gravity and scalar-tensor gravity theories, while they do not appear in Einstein and Lovelock gravity theories etc. With the general procedure, we will study the thermodynamics in the Randall-Sundrum braneworld imbedded in the nontrivial bulk. Recently, the model has been used to mimic the phantom-like dark energy without violating weak energy condition [14, 15, 16, 17].

In [18], it was argued that there is a mass-like function connecting the first law of thermodynamics and the Friedmann equations in some gravity theories. It is of interest to investigate whether this mass-like function has the connection with the Misner-Sharp mass. The Misner-Sharp mass is widely accepted as a quasi-local mass in Einstein gravity and its generalized forms have been obtained in Gauss-Bonnet gravity, Lovelock gravity, etc [19, 20]. We will show that in Lovelock gravity, the generalized Misner-Sharp mass is a special form of the mass-like function.

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The organization of the paper is the following: In section 2, we will present the general procedure to construct the first law of thermodynamics on the apparent horizon. In the following section we will examine this procedure and show that it is valid by illustrating it to some extended gravity theories including Lovelock,  $f(R)$  and scalar-tensor gravity theories. In section 4, we will apply the general procedure to study the thermodynamics on the apparent horizon in Randall-Sundrum braneworld imbedded in the nontrivial bulk. Our conclusion and discussion will be present in the last section.

## II. THE FIRST LAW OF THERMODYNAMICS ON APPARENT HORIZON OF FRW COSMOLOGY

In this section, we are going to present a general procedure to construct the first law of thermodynamics on the apparent horizon. We will consider extended gravity theories based on Einstein gravity.

The homogenous and isotropic  $(n+1)$ -dimensional FRW universe is described by

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2, \quad (1)$$

where  $h_{ab} = \text{diag}(-1, \frac{a^2}{1-ka^2})$ ,  $d\Omega_{n-1}^2$  is the  $(n-1)$ -dimensional sphere element, and  $x^0 = t$ ,  $x^1 = r$ ,  $\tilde{r} = ar$  is the radius of the sphere and  $a$  is the scale factor. For simplicity, we consider the flat space  $k = 0$  in this paper, however, our discussion can also be generalized to the non-flat cases. It is known that the dynamical apparent horizon, the marginally trapped surface with vanishing expansion, is defined as a sphere situated at  $r = r_A$  satisfying

$$h^{ab}\partial_a \tilde{r} \partial_b \tilde{r} = 0.$$

The sphere radius is

$$\tilde{r}_A \equiv r_A a = \frac{1}{H}. \quad (2)$$

The associated temperature on the apparent horizon can be defined as

$$T = \frac{1}{2\pi\tilde{r}_A}. \quad (3)$$

In Einstein gravity, the entropy is proportional to the horizon area

$$S_E = \frac{A}{4G},$$

where the horizon area  $A = n\Omega_n \tilde{r}_A^{n-1}$ , thus we have the fundamental relation

$$\delta Q \equiv T dS_E = \frac{n(n-1)V\tilde{r}_A^{-3}d\tilde{r}_A}{8\pi G},$$

where  $V = \Omega_n \tilde{r}_A^n$  is the volume in the horizon. Using the definition of the horizon (2), we can obtain

$$T dS_E = \frac{-n(n-1)V}{16\pi G} \frac{dH^2}{dt} dt, \quad (4)$$

which is purely a geometric relation.

For all gravity theories, one can write the Friedmann equations as the form of Einstein gravity

$$H^2 = \frac{16\pi G}{n(n-1)} \rho_{eff} \quad (5)$$

$$\dot{H} = -\frac{8\pi G}{(n-1)} (\rho_{eff} + p_{eff}). \quad (6)$$

Though we do not know the exact form of  $\rho_{eff}$  (and  $p_{eff}$ ), we know that there must be ordinary matter density  $\rho$  in  $\rho_{eff}$  and also other quantities  $\rho_i$ , such as matter or energy components besides the ordinary matter. The first Friedmann equation can be expressed in the form

$$H^2 = H^2(\rho, \rho_1, \dots, \rho_i, \dots).$$

Then the relation (4) can be changed as

$$TdS_E = \frac{-n(n-1)V}{16\pi G} dt \left( \frac{\partial H^2}{\partial \rho} \dot{\rho} + \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \right). \quad (7)$$

To construct the first law of thermodynamics  $dE = TdS$ , we need to know the energy flux  $dE$  and entropy  $S$ . In the general gravity theory, they are not specified. The energy flux of ordinary matter can be expressed as  $dE = V\dot{\rho}dt$ . Multiplying  $\frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}}$  on both sides of (7), we have

$$\frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} TdS_E = -V\dot{\rho}dt - Vdt \frac{1}{\frac{\partial H^2}{\partial \rho}} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i. \quad (8)$$

In the general case, the conservation of the total matter density can be written as

$$\dot{\rho}_{eff} + nH(\rho_{eff} + p_{eff}) = 0, \quad (9)$$

while we assume that ordinary matter has energy exchange  $q$  with other matter or energy contents,

$$\dot{\rho} + nH(\rho + p) = q. \quad (10)$$

Equations (9) and (10) will be used to express the first law explicitly. Substituting (10), (8) can be changed to the form

$$T \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} dS_E = nVH(\rho + p)dt - Vqdt - Vdt \frac{1}{\frac{\partial H^2}{\partial \rho}} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i. \quad (11)$$

The entropy form can be got by integrating (11). If there is just ordinary matter  $\rho$  in the space,  $\frac{\partial H^2}{\partial \rho}$  can be rewritten as a function of  $\tilde{r}_A$ . Then the entropy can be obtained by the integration

$$\begin{aligned} S &= \int \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} d(S_E) \\ &= \int \frac{4\pi \tilde{r}_A^{n-2} \Omega_n}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} d\tilde{r}_A \end{aligned} \quad (12)$$

and the relation (11) can be written as

$$TdS = dE, \quad (13)$$

where

$$dE = V\dot{\rho}dt = nVH(\rho + p)dt.$$

It is the first law of thermodynamics for the gravity theories with only freedom  $\rho$  in the first Friedmann equation.

If the extended gravity theory has other dynamic fields resulting that  $\frac{\partial H^2}{\partial \rho}$  is a function of  $\tilde{r}_A$  and  $\rho_i$

$$\frac{\partial H^2}{\partial \rho} = \frac{\partial H^2}{\partial \rho}(\tilde{r}_A, \rho_i).$$

The last term on the r.h.s in Eq. (11) can not be included in the total differential in general. However we can express the l.h.s in (11) in the form

$$T \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} dS_E = Td \left( \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} S_E \right) - T \frac{16\pi G}{n(n-1)} S_E d \frac{1}{\frac{\partial H^2}{\partial \rho}}. \quad (14)$$

Then the general expression of the first law of thermodynamics for generalized gravity theories with more freedoms in the first Friedmann equation reads

$$TdS + Td_i S = dE. \quad (15)$$

$d_i S$  is defined as

$$d_i S \equiv -\frac{16\pi G}{n(n-1)} S_E d\frac{1}{\frac{\partial H^2}{\partial \rho}} = -\frac{4\pi \tilde{r}_A^{n-1} \Omega_n}{(n-1)} d\frac{1}{\frac{\partial H^2}{\partial \rho}}, \quad (16)$$

which is interesting since it relates to the entropy production in the nonequilibrium thermodynamics. This entropy production term comes because it cannot be absorbed in the complete derivative as in the usual gravity theory. The energy flux is defined as

$$dE \equiv -V \dot{\rho} dt - V dt \frac{1}{\frac{\partial H^2}{\partial \rho}} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i = nVH(\rho + p)dt - V q dt - V dt \frac{1}{\frac{\partial H^2}{\partial \rho}} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i. \quad (17)$$

The entropy can be obtained as

$$S = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} S_E = \frac{4\pi \tilde{r}_A^{n-1} \Omega_n}{(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}}. \quad (18)$$

The exact form of entropy production  $d_i S$ , energy flux  $dE$  and entropy  $S$  depend on the corresponding gravity theory.

Recently, a generalized mass-like function in  $(3+1)$ -dimensional Einstein gravity

$$M \equiv \frac{\tilde{r}}{2G} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}) \quad (19)$$

has been used to connect the Friedmann equations and the first law of thermodynamics on the apparent horizon [18]

$$TdS_E = \frac{1}{G} d\tilde{r}_A = k^a \nabla_a M dt = 4\pi \tilde{r}_A^3 H(\rho + p) dt = dE. \quad (20)$$

The mass formulae

$$M_{,a} = -4\pi \tilde{r}^2 (T_a^b - \delta_a^b T) \tilde{r}_{,b} + \tilde{r}_{,a}$$

has been used to derive the third equality in (20). But whether this mass-like function is general and valid is not clear. Here we will address this point.

The essence of relation (20) is to introduce the  $M$  satisfying the geometric relation

$$TdS_E = k^a \nabla_a M dt$$

where  $k^a$  is the (approximate) generator  $k^a = (1, -rH)$  of the apparent horizon which is null at the horizon. The third equality of relation (20) can be proved by Friedmann equations (5) and (6), even without introducing the mass formulae in (20). To derive the generalized mass-like function, one must know the entropy expression at first. Substituting the entropy expression (12) we have equalities on the horizon

$$TdS = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} TdS_E = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} k^a \nabla_a [M + f_1] dt = k^a \nabla_a \tilde{M} dt, \quad (21)$$

where  $M$  is the  $(n+1)$ -dimensional generalization of the mass-like function (19) in  $(3+1)$  dimensions

$$M = \frac{n(n-1)\Omega_n \tilde{r}^{n-2}}{16\pi G} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}).$$

Noting  $\frac{\partial H^2}{\partial \rho}$  is just a function of  $\tilde{r}_A$ , using the constraint

$$k^a \tilde{r}_{,a} = 0, \quad (22)$$

and remembering that  $f_i$  ( $i = 1, 2$ ) are arbitrary functions satisfying

$$k^a f_{i,a} = 0 \quad (23)$$

on the horizon, we can get the last equality of (21) provided that

$$\tilde{M} = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} [M + f_1] + f_2. \quad (24)$$

The mass-like function  $\tilde{M}$  is determined up to functions  $f_i$ , which means that  $\tilde{M}$  can have many desired properties in different cases by tuning the functions  $f_i$ . For the most simplest case, the Einstein gravity, if we set  $f_1 = -\frac{2n(n-1)\Omega_n \tilde{r}_A^{n-2}}{16\pi G}$  and  $f_2 = 0$ , the mass function  $\tilde{M}$  is just the negative Misner-Sharp mass

$$\tilde{M} = -M_{ms} = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2}}{16\pi G} (-1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}).$$

The situation is different for the gravity theories with more matter or energy components. Using the entropy expression (18), we have the following equation at the horizon

$$\begin{aligned} TdS &= \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A, \rho_i(t))} TdS_E + \frac{n\Omega_n \tilde{r}_A^{n-2}}{8\pi G} d\left(\frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A, \rho_i(t))}\right) \\ &= \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A, \rho_i(t))} k^a \nabla_a (M + f_3) dt + \frac{2}{(n-1)} (M + f_3) d\left(\frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A, \rho_i(t))}\right) \\ &= k^a \nabla_a \tilde{M} dt, \end{aligned}$$

where  $f_3$  satisfies

$$k^a f_{3,a} = 0,$$

and  $f_3 = 0$  at the horizon. Noting now  $\frac{\partial H^2}{\partial \rho}$  is a function of  $\tilde{r}_A$  and  $\rho_i$ , and using the constraint (22), we finds that only when  $n = 3$ , the last equation has the general solution

$$\tilde{M} = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} (M + f_3) + f_2, \quad n = 3. \quad (25)$$

Setting  $f_1 = f_3$ , expressions of mass-like function in (24) and (25) are the same when  $n = 3$ . The mass-like function  $\tilde{M}$  embodies the specific property in the  $(3+1)$ -dimensional gravity.

### III. THERMODYNAMICS OF SOME EXTENDED GRAVITY THEORIES

In this section, we will calculate the entropy expressions (12) (18) and give explicitly the first law of thermodynamics (13) (15) in some extended gravity theories, including the Lovelock,  $f(R)$  and scalar-tensor gravity theories. We will give explicitly the mass-like functions accordingly.

#### A. Lovelock gravity

The Lagrangian of the Lovelock gravity [24] consists of the dimensionally extended Euler densities

$$L = \sum_{i=1}^{[n/2]} c_i L_i,$$

where  $c_i$  is an arbitrary constant and  $L_i$  is the Euler density of a  $(2i)$ -dimensional manifold

$$L_i = 2^{-i} \delta^{\mu_1 \nu_1 \dots \mu_i \nu_i}_{\alpha_1 \beta_1 \dots \alpha_i \beta_i} R_{\mu_1 \nu_1 \dots \mu_i \nu_i}^{\alpha_1 \beta_1 \dots \alpha_i \beta_i}.$$

$L_1$  is just the Einstein-Hilbert term, and  $L_2$  corresponds to the so called Gauss-Bonnet term. Using the FRW metric, we obtain the Friedmann equations in  $(n+1)$ -dimensional space-time

$$\sum_{i=1}^{[n/2]} \hat{c}_i (H^2)^i = \frac{16\pi G}{n(n-1)} \rho, \quad (26)$$

and

$$\sum_{i=1}^{[n/2]} \hat{c}_i i (H^2)^{i-1} (\dot{H}) = -\frac{8\pi G}{(n-1)}(\rho + p),$$

where

$$\hat{c}_i = \frac{(n-2)!}{(n-2i)!} c_i$$

Since there is only one freedom  $\rho$  in the first Friedmann equation (26), we will use the entropy expression (12), and obtain

$$\begin{aligned} S &= \int \frac{4\pi \tilde{r}_A^{n-2} \Omega_n}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} d\tilde{r}_A \\ &= \frac{n\Omega_n}{4G} \sum_{i=1}^{[n/2]} \frac{i(n-1)}{n-2i+1} \hat{c}_i \tilde{r}_A^{n-2i+1}. \end{aligned} \quad (27)$$

and the corresponding first law (13) reads

$$TdS = dE = n\Omega_n \tilde{r}_A^n H(\rho + p),$$

where the second equality holds since there is not  $q$  and  $\rho_i$ .

The mass-like function (24) now reads

$$\begin{aligned} \tilde{M} &= \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} [M + f_1] + f_2 \\ &= \frac{n(n-1)\Omega_n \tilde{r}_A^n}{16\pi G} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}) \sum_{i=1}^{[n/2]} \hat{c}_i i \tilde{r}_A^{-2i} + f_1 \sum_{i=1}^{[n/2]} \hat{c}_i i \tilde{r}_A^{-2i} + f_2. \end{aligned} \quad (28)$$

In Ref. [19], the generalized Misner-Sharp mass for Lovelock gravity was conjectured

$$M_{ms} = \frac{n(n-1)\Omega_n \tilde{r}_A^n}{16\pi G} \sum_{i=1}^{[n/2]} \hat{c}_i \tilde{r}_A^{-2i} (1 - h^{ab} \tilde{r}_{,a} \tilde{r}_{,b})^i. \quad (29)$$

Setting

$$\begin{aligned} f_1 &= -2 \frac{n(n-1)\Omega_n \tilde{r}_A^n}{16\pi G} \\ f_2 &= \frac{n(n-1)\Omega_n \tilde{r}_A^n}{16\pi G} \sum_{i=1}^{[n/2]} \hat{c}_i \tilde{r}_A^{-2i} [i(1 - h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}) - (1 - h^{ab} \tilde{r}_{,a} \tilde{r}_{,b})^i], \end{aligned}$$

which satisfy (23), we can easily find that  $M_{ms}$  in (29) is the same as negative (28). Thus, the generalized Misner-Sharp mass can be considered as a specific case of the (negative) mass function  $\tilde{M}$ .

## B. Nonlinear gravity

For the nonlinear gravity  $f(R)$ , the Lagrangian is

$$L = \frac{1}{16\pi G} f(R)$$

The variational principle gives equations of motion. Using the FRW metric, one can obtain the Friedmann equations in  $(n + 1)$ -dimensional space-time

$$H^2 = \frac{16\pi G}{n(n-1)} \frac{1}{f'} (\rho + \rho_c f') \quad (30)$$

$$\dot{H} = -\frac{8\pi G}{(n-1)} \frac{1}{f'} (\rho + \rho_c f' + p + p_c f'), \quad (31)$$

where

$$\rho_c = \frac{1}{8\pi G f'} \left[ -\frac{f - Rf'}{2} - nH f'' \dot{R} \right]$$

$$p_c = \frac{1}{8\pi G f'} \left[ (f - Rf') - f'' \ddot{R} + f''' \dot{R}^2 + n(n-1) f'' \dot{R} \right].$$

There are other freedoms besides  $\rho$  in the first Friedmann equation (30), so one should consider the non-equilibrium thermodynamics. One can select  $\rho_i$  arbitrarily. For example, we select  $\rho_i = (f', \rho_c)$ . But it should be noticed that there is not real matter fields besides the ordinary matter  $\rho$ . From the first Friedmann equation (30), one can explain  $\rho_e \equiv \rho_c f'$  ( $p_e \equiv p_c f'$ ) as the density (pressure) of an effective energy component in  $f(R)$  gravity. Now recalling the conserved equation (9)

$$\dot{\rho}_{eff} = -nH(\rho_{eff} + p_{eff}) \quad (32)$$

and using the first Friedmann equation (30), one can find that the l.h.s in Eq. (32) reads

$$\begin{aligned} \dot{\rho}_{eff} &= \frac{n(n-1)}{16\pi G} \left( \frac{\partial H^2}{\partial \rho} \dot{\rho} + \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \right) \\ &= \frac{n(n-1)}{16\pi G} \left( \frac{16\pi G}{n(n-1)} \frac{1}{f'} \dot{\rho} + \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \right) \\ &= \frac{1}{f'} \dot{\rho} + \frac{n(n-1)}{16\pi G} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \end{aligned} \quad (33)$$

while the r.h.s in Eq. (32) reads by the second Friedmann equation (31)

$$-nH(\rho_{eff} + p_{eff}) = -nH \left[ \frac{1}{f'} (\rho + p + \rho_e + p_e) \right]. \quad (34)$$

Employing the continuous equation (10) to Eqs. (33) and (34), one can find

$$\frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i = -\frac{16\pi G}{(n-1)} \frac{1}{f'} [H(\rho_e + p_e) + q]. \quad (35)$$

By substituting Eq. (35) into Eq. (17), we have

$$\begin{aligned} dE &= nVH(\rho + p)dt - Vqdt - Vdt \frac{1}{\frac{\partial H^2}{\partial \rho}} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \\ &= nVH(\rho + p)dt + nVH(\rho_e + p_e)dt. \end{aligned} \quad (36)$$

Then we have the first law

$$TdS + Td_iS = dE,$$

where the entropy production and entropy can be obtained from Eq. (16) and Eq. (18) respectively

$$d_iS = -S_E df', \quad (37)$$

and

$$S = S_E f' = \frac{A}{4G} f'. \quad (38)$$

It should be noticed that the energy flux in Eq. (36) is different with the definition in [7, 8]. But one can find that it is indeed originated from the energy flux of matter and effective energy component in  $f(R)$  gravity.

Setting  $f_2 = f_3 = 0$ , the mass-like function (25) for  $n = 3$  is

$$\begin{aligned} \tilde{M} &= \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} M \\ &= \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2}}{16\pi G} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}) f'. \end{aligned} \quad (39)$$

### C. Scalar-tensor gravity

The general scalar-tensor theory of gravity is described by the Lagrangian

$$L = \frac{1}{16\pi G} F(\phi) R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi),$$

where  $F(\phi)$  is a positive continuous function of the scalar field  $\phi$  and  $V(\phi)$  is its potential. Using the FRW metric, we obtain the Friedmann equations in  $(n+1)$ -dimensional space-time

$$H^2 = \frac{16\pi G}{n(n-1)} \frac{1}{F} (\rho + \rho_f + \rho_c F) \quad (40)$$

$$\dot{H} = \frac{8\pi G}{(n-1)} \frac{1}{F} (\rho + p + \rho_f + p_f + \rho_c F + p_c F). \quad (41)$$

where the density and pressure of scalar field  $\phi$  are

$$\begin{aligned} \rho_f &= \frac{1}{2} \dot{\phi}^2 + V(\phi) \\ p_f &= \frac{1}{2} \dot{\phi}^2 - V(\phi), \end{aligned}$$

and  $\rho_e \equiv \rho_c F$  ( $p_e \equiv p_c F$ ) may be understood as effective density (pressure) of the energy component in scalar-tensor theory:

$$\rho_c = -\frac{n}{8\pi G F} H \dot{F}$$

$$p_c = \frac{1}{8\pi G F} [\ddot{F} + (n-1) H \dot{F}].$$

Obviously, we need to consider the non-equilibrium thermodynamics. We select  $\rho_i = (\rho_f, F, \rho_e)$ . Now recalling the continuous equation (9) and using the first Friedmann equation (30), one can find that the l.h.s in Eq. (32) reads

$$\begin{aligned} \dot{\rho}_{eff} &= \frac{n(n-1)}{16\pi G} \left( \frac{\partial H^2}{\partial \rho} \dot{\rho} + \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \right) \\ &= \frac{n(n-1)}{16\pi G} \left[ \frac{16\pi G}{n(n-1)} \frac{1}{F} \dot{\rho} + \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \right] \\ &= \frac{1}{F} \dot{\rho} + \frac{n(n-1)}{16\pi G} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \end{aligned} \quad (42)$$

while the r.h.s in Eq. (32) reads by the second Friedmann equation (41)

$$-nH(\rho_{eff} + p_{eff}) = -nH \frac{1}{F} (\rho + p + \rho_f + p_f + \rho_e + p_e). \quad (43)$$



Employing the continuous equation (10) to Eqs. (42) and (43), one can find

$$\frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i = -\frac{16\pi G}{(n-1)} \frac{1}{F} [H(\rho_f + p_f + \rho_e + p_e) + q]. \quad (44)$$

By substituting Eq. (44) into Eq. (17), we obtain

$$\begin{aligned} dE &= nVH(\rho + p)dt - Vqdt - Vdt \frac{1}{\frac{\partial H^2}{\partial \rho}} \frac{\partial H^2}{\partial \rho_i} \dot{\rho}_i \\ &= nVH(\rho + p + \rho_f + p_f)dt + nVH(\rho_e + p_e)dt, \end{aligned} \quad (45)$$

where  $nVH(\rho_e + p_e)dt$  denotes the energy flux of effective energy component  $\rho_e$  in scalar-tensor gravity. Then we have the first law

$$TdS + Td_iS = dE,$$

where the entropy production and entropy can be obtained from Eq. (16) and Eq. (18) respectively

$$d_iS = -S_E dF,$$

and

$$S = S_E F = \frac{A}{4G} F. \quad (46)$$

Setting  $f_2 = f_3 = 0$ , the mass-like function (25) for  $n = 3$  is

$$\tilde{M} = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2}}{16\pi G} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}) F. \quad (47)$$

It is interesting to note that entropy expressions (27), (38), and (46) are consistent with black hole entropy of Lovelock gravity [21], nonlinear gravity [22] and scalar-tensor gravity [23], respectively. The obtained mass-like functions (28), (39), and (47) also agree to those presented in [18].

#### IV. THERMODYNAMICS OF RANDALL-SUNDRUM BRANEWORLD WITH NONTRIVIAL BULK

We consider a  $n$ -dimensional brane embedded in a  $(n+2)$ -dimensional space-time. For convenience and without loss of generality we choose the extra dimension along the coordinates  $y$  such that the brane is located at  $y = 0$  and the bulk has  $Z_2$  symmetry under the transformation  $y \rightarrow -y$ . The action is

$$S = \frac{1}{2\kappa_5^2} \int d^{n+2}x \sqrt{-g} \{\mathcal{L}_{EH}\} + \int d^{n+2}x \sqrt{-g} \{\mathcal{L}_M\} + \int d^{n+1}x \sqrt{-\tilde{g}} \{\mathcal{L}_m - 2\lambda\}, \quad (48)$$

where  $g$  ( $\tilde{g}$ ),  $\kappa_{n+2}$  ( $\kappa_{n+1}$ ), and  $\mathcal{L}_M$  ( $\mathcal{L}_m$ ) are the bulk (brane) metric, bulk (brane) gravitational constant, and bulk (brane) matter fields, respectively.  $\mathcal{L}_{EH} = R - 2\Lambda$  is the five-dimensional Einstein-Hilbert Lagrangian with negative cosmological constant  $\Lambda < 0$ .  $\lambda$  is the brane tension (or cosmological constant). For convenience, we will choose the unit that  $\kappa_{n+2} = 1$ .

By varying the action in Eq. (48) with respect to the bulk metric, we obtain the field equation

$$G_{AB} = T_{AB}|_{total}, \quad (49)$$

The total energy-momentum tensor  $T_{AB}|_{total}$  is decomposed into bulk and brane components

$$T_{AB}|_{total} = T_{AB}|_{bulk} + T_{AB}|_{brane} \hat{\delta}(y).$$

Here, we use the normalized Dirac delta function,  $\hat{\delta}(y) = \sqrt{\tilde{g}/g} \delta(y)$ . The bulk component is

$$T_{AB}|_{bulk} = -\Lambda g_{AB} + T_{AB},$$

where  $T_{AB}$  denotes all possible energy-momentum in the bulk. The brane component is written as

$$T_{AB}|_{brane} = -\lambda\tilde{g}_{AB} + \tilde{T}_{AB},$$

where energy momentum tensor  $\tilde{T}_{AB}$  represents matter on the brane with energy density  $\rho$  and pressure  $p$ .

The  $(n+2)$ -dimensional line element in the bulk is given by

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\gamma_{ij}dx^i dx^j + b^2(t, y)dy^2$$

where  $\gamma_{ij}$  is a  $n$ -dimensional maximally symmetric metric whose spatial curvature is characterized by  $k = 0, \pm 1$ . Here we are interested in spatially flat brane  $k = 0$ . We choose the coefficients  $n(t, 0) = 1$  so that  $t$  is the proper time along the brane. For simplicity, we assume that the fifth dimension is static  $\dot{b} = 0$  and we set  $b = 1$ .

To determine the Friedmann equations on the brane, we need the junction condition on the brane. For convenience, we use prime and dot to denote the derivative with respect to  $y$  and  $t$ , respectively. The jump of the (00) and  $(ij)$  components of the field equation (49) across the brane gives

$$\frac{a'_+}{a_0} = -\frac{1}{2n}(\rho + \lambda), \quad (50)$$

$$\frac{n'_+}{n_0} = \frac{p + \frac{n-1}{n}\rho}{2}, \quad (51)$$

where  $2a'_+ = -2a'_-$  and  $2n'_+ = -2n'_-$  are the discontinuities of the first derivatives. Substituting Eq. (50) and Eq. (51) into the (05) component of the field equation (49)

$$n\left(\frac{n'}{n}\frac{\dot{a}}{a} - \frac{\dot{a}'}{a}\right) = T_{05}, \quad (52)$$

we obtain the continuity equation

$$\dot{\rho} + nH(\rho + p) = 2T_{05}. \quad (53)$$

From the (00) and (55) equations, following [25], we find a set of functions  $a(t, y)$  and  $n(t, y)$

$$\Phi = \left(\frac{\dot{a}}{na}\right)^2 - \frac{a'^2}{a^2} - \frac{2\Lambda_{n+2}}{n(n+1)}. \quad (54)$$

When the bulk is  $\text{AdS}_5$ , it satisfies the constraint equation

$$\Phi = 0.$$

But for nontrivial bulk,  $\Phi$  will not be a constant. Even when  $T_{05} = T_{55} = 0$ , a weyl radiation term  $\sim a^{-n-1}$  may appear. One can find the evolved equation of  $\Phi$  in [26], and the physical meaning of  $\Phi$  is the correction to the bulk cosmological constant. Substituting the junction conditions (50) and (51) into Eq. (54), we can obtain the first Friedmann equation

$$H^2 = \frac{1}{4n^2}\rho^2 + \frac{1}{2n^2}\lambda\rho + \Phi \quad (55)$$

where the Randall-Sundrum fine-tuning condition

$$\frac{1}{4n^2}\lambda^2 + \frac{2\Lambda_{n+2}}{n(n+1)} = 0$$

has been used.

There are other freedoms besides  $\rho$  in the first Friedmann equation (55), so one should use the entropy expression (18). We select  $\rho_1 = \Phi$ . Now Eq. (17) reads

$$\begin{aligned} dE &= nVH(\rho + p)dt - Vqdt - Vdt\frac{1}{\frac{\partial H^2}{\partial \rho}}\frac{\partial H^2}{\partial \rho_i}\dot{\rho}_i \\ &= [nVH(\rho + p)dt - 2VT_{05}dt] - Vdt\frac{2n^2\tilde{r}_A}{\sqrt{\tilde{r}_A^2\lambda^2 - 4n^2(\tilde{r}_A^2\Phi - 1)}}\dot{\Phi}, \end{aligned}$$

Then we have the first law

$$TdS + Td_iS = dE,$$

where the entropy production and entropy can be obtained from Eq. (16) and Eq. (18) respectively

$$d_iS = \frac{8\pi n^2 \tilde{r}_A^{n-1} \Omega_n}{(n-1)} d \frac{\tilde{r}_A}{\sqrt{\tilde{r}_A^2 \lambda^2 - 4n^2 (\tilde{r}_A^2 \Phi - 1)}}$$

and

$$S = \frac{8\pi n^2 \Omega_n \tilde{r}_A^n}{(n-1) \sqrt{\lambda^2 - 4n^2 \tilde{r}_A^2 (\Phi - 1)}}. \quad (56)$$

The mass-like function (25) for  $n = 3$  is now expressed as

$$\tilde{M} = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} M = \frac{2n^2 \Omega_n \tilde{r}_A^{n-1}}{\sqrt{\tilde{r}_A^2 \lambda^2 - 4n^2 (\tilde{r}_A^2 \Phi - 1)}} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}).$$

To compare with the result obtained in [10], we set  $\Phi = 0$ . Now the entropy expression (12) should be used,

$$S = \int \frac{4\pi \tilde{r}_A^{n-2} \Omega_n}{\frac{\partial H^2}{\partial \rho}(\tilde{r}_A)} d\tilde{r}_A = \int \frac{8\pi n^2 \Omega_n \tilde{r}_A^{n-1}}{\sqrt{\tilde{r}_A^2 \lambda^2 + 4n^2}} d\tilde{r}_A. \quad (57)$$

One can find that the entropy expression is consistent with the result obtained in [10]. The corresponding first law (13) reads

$$TdS = dE = -V\dot{\rho}dt = n\Omega_n \tilde{r}_A^n H(\rho + p),$$

where Eq. (17) without  $q$  and  $\rho_i$  has been used to get the second equality.

Setting  $f_2 = f_3 = 0$ , the mass-like function (24) now reads

$$\tilde{M} = \frac{16\pi G}{n(n-1)} \frac{1}{\frac{\partial H^2}{\partial \rho}} M = \frac{2n^2 \Omega_n^2 \tilde{r}_A^{2n-3}}{\sqrt{\tilde{r}_A^2 \lambda^2 + 4n^2}} (1 + h^{ab} \tilde{r}_{,a} \tilde{r}_{,b}).$$

## V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have constructed the first law of thermodynamics on the apparent horizon of generalized gravity theories, including Einstein gravity, Lovelock,  $f(R)$  and scalar-tensor gravity theories. We have also generalized our study to the Randall-Sundrum braneworld with nontrivial bulk and obtained the corresponding entropy and the first law of thermodynamics. It is interesting to observe that entropy expressions (12) and (18) are general, which can lead to the consistency with the black hole entropy obtained in extended gravity theories such as Lovelock,  $f(R)$  and scalar-tensor gravity theories. This gives us the hope that the general entropy expression can be used to shed some lights on even more generalized gravity theory such as the Randall-Sundrum braneworld etc., where until now there is no exact black hole solution obtained.

One may argue that there seems to be ambiguities in the entropy expressions (12) (18), since one might add proper quantity to the expression of entropy, which may vanish in the Einstein gravity, and this extra term could be absorbed into the redefinition of the entropy production. This worry is not necessary, since the known black hole entropy in different gravity theories will strictly restrict the form of the additional quantities in the entropy expressions. Entropy expressions (27) (38) and (46) are consistent with the known black hole entropy in Lovelock gravity [21], nonlinear gravity [22] and scalar-tensor gravity [23], respectively. This fact shows that it is not needed to add additional quantities to the entropy expressions. For the braneworld case, without adding additional terms, (57) reduces to the area formula in  $n + 1$  dimensions in the large horizon limit, while in the small horizon limit, it becomes the area formula in the bulk [10,27]. On the other hand, we have built the relation between the general mass-like functions and the entropy expressions. The obtained mass-like functions (28) (39) and (47) are in agreement with those presented in [18]. The general mass-like functions have dimension of energy. However if one adds other quantities in (12) (18), the derived mass-like function cannot reduce to that obtained in [18]. This serves as another restriction on adding additional terms to the entropy expressions.

Our formalism of constructing the first law of thermodynamics is general and can be applied to any gravity theory no matter matter contents are conserved or not. We find that the non-equilibrium entropy production appears due to the other dynamic fields besides the ordinary matter dominating the cosmological evolution.

We have argued that the mass-like function presented in [18] is general in extended gravity theories. In Lovelock gravity, the conjectured generalized Misner-Sharp mass in [20] is the special case of the mass-like function. This sparks us to further investigate the physical meaning of the mass-like function and its relation to the generalized Misner-Sharp mass in generalized gravity theories.

### Acknowledgments

This work was partially supported by the NSFC, Shanghai Education Commission, Science and Technology Commission. S. F. Wu and G. H. Yang were also supported by the NSFC under Grant No. 10575068, the Shanghai Education Development Foundation, and the Natural Science Foundation of Shanghai Municipal Science Technology Commission under grant Nos. 04ZR14059 and 04dz05905.

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